Heavy quarkonium according to resummed perturbation theory

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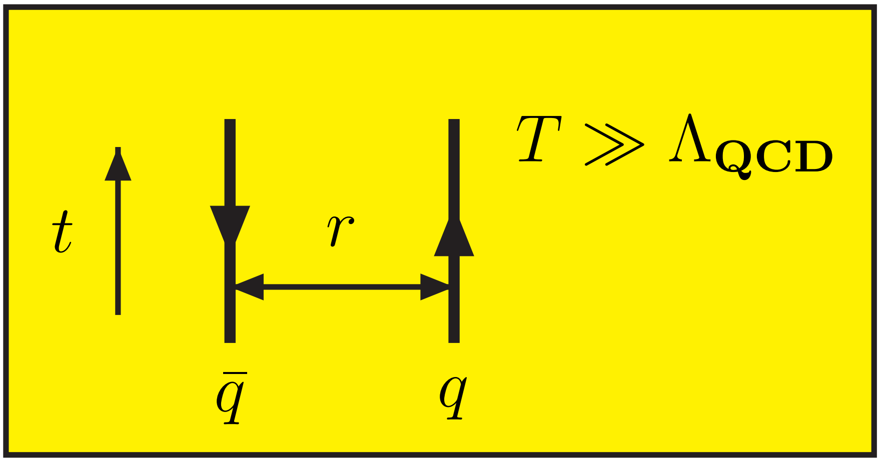
Miscellaneous remarks on:

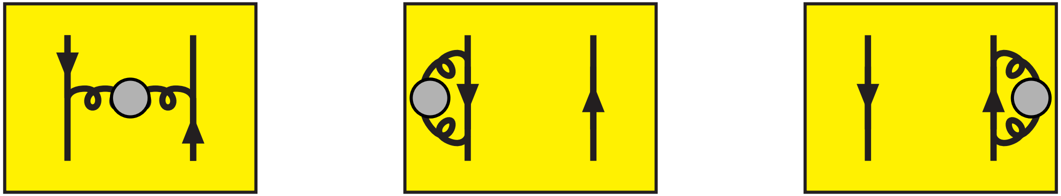
1. Real-time static potential at finite temperature
2. Relation of static potential and quarkonium spectral function
3. Physics lessons for the dilepton production rate
4. Mystery with the scalar channel

1. Definition of a real-time static potential

Physical picture: an infinitely heavy quark–antiquark pair propagates in Minkowski time within a thermalized QCD medium.



At weak coupling:



What is the static limit of the time-ordered HTL-resummed gluon propagator in **real Minkowski time**?

$$iD_{00}^{T}(0,\mathbf{q}) = \frac{1}{\mathbf{q}^2 + m_{\mathrm{D}}^2} - i\frac{\pi m_{\mathrm{D}}^2 T}{|\mathbf{q}|(\mathbf{q}^2 + m_{\mathrm{D}}^2)^2}\,.$$

Beraudo et al, 0712.4394

More concretely, consider the Schrödinger eqn satisfied by a suitably defined Green's function,

$$i\partial_t C_>(t,r) \equiv [2M + V_>(t,r)]C_>(t,r) \ ,$$

or the analytic continuation $\tau \to it$ of a Euclidean Wilson loop $W_E(\tau,r)$,

$$i\partial_t W_E(it,r) \equiv V_>(t,r) W_E(it,r) \ .$$

To $\mathcal{O}(g^2)$, both yield:

$$V_>(\infty,r) = g^2 C_F \int \frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^3} \left(1 - e^{i\mathbf{q}\cdot\mathbf{r}}\right) \times iD^T_{00}(0,\mathbf{q}) \ .$$

So, there is a complex potential!

ML et al, hep-ph/0611300

$$\mathrm{Re}\, V_>(\infty, r) \;=\; -\frac{g^2 C_F}{4\pi}\left[m_\mathrm{D} + \frac{\exp(-m_\mathrm{D} r)}{r}\right] \;,$$

$$\mathrm{Im}\, V_>(\infty, r) \;=\; -\frac{g^2 T C_F}{4\pi}\,\phi(m_\mathrm{D} r) \;,$$

where

$$\phi(x) = 2\int_0^\infty \frac{\mathrm{d}z\, z}{(z^2+1)^2}\left[1 - \frac{\sin(zx)}{zx}\right] \;,$$

is finite and strictly increasing, with the limiting values $\phi(0) = 0$, $\phi(\infty) = 1$.

Physics interpretation of the real part at $r \to \infty$:

$2 \times$ thermal mass correction for a heavy quark.

Physics interpretation of the imaginary part at $r \to \infty$:

Beraudo et al, 0712.4394

$2 \times$ thermal decay width of a heavy quark.

Pisarski, PRL 63 (1989) 1129

To summarize:

There is a non-vanishing imaginary part in the real-time static potential.

It isn't parametrically suppressed (in g^2 or N_c).

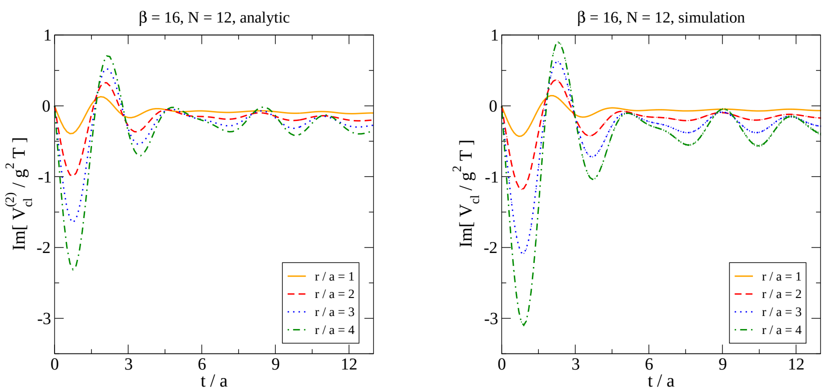
Its physics is not specific to weak coupling.

Can one measure $V_>$ non-perturbatively?

If keep $\hbar \neq 1$, then $g^2 \to g^2\hbar$ and $\beta \to \beta\hbar$. In the classical limit $\hbar \to 0$, the real part $\sim g^2\hbar/r$ disappears, but the imaginary part $\sim g^2T = g^2\hbar/\beta\hbar$ survives.

The im. part can thus be measured with classical lattice gauge theory simulations, just like the sphaleron rate.

ML et al, 0707.2458



Open: non-perturbative real part of $V_>(\infty, r)$

It does have the correct short-distance behaviour
$\Rightarrow$ not $\langle \mathrm{Tr}[P(0)]\,\mathrm{Tr}[P^\dagger(\mathbf{r})] \rangle$.

It is an explicitly gauge invariant function of r
$\Rightarrow$ probably not Coulomb gauge $\langle \mathrm{Tr}[P(0)P^\dagger(\mathbf{r})] \rangle$.

What is it then?

(There is also a lot of work on the static potential in AdS/CFT [0803.3070 and refs therein], however its precise relation to the present work is not clear. In particular, the computations so far yielded no imaginary part [refined computations could allegedly do this].)

2. From static potential to heavy quarkonium

Dilepton production rate:

$$\frac{\mathrm{d}N_{\mu^+\mu^-}}{\mathrm{d}^4x\mathrm{d}^4Q} = -\frac{e^2}{3(2\pi)^5Q^2}\Big(1+\frac{2m_\mu^2}{Q^2}\Big)\Big(1-\frac{4m_\mu^2}{Q^2}\Big)^{\frac{1}{2}} e^{-\frac{q^0}{T}}\tilde{C}_>(Q) \;,$$

$$\tilde{C}_>(Q) \equiv \int_{-\infty}^{\infty}\mathrm{d}t \int \mathrm{d}^3\mathbf{x}\, e^{iQ\cdot x}\langle \hat{\mathcal{J}}^\mu(x)\hat{\mathcal{J}}_\mu(0)\rangle \;,$$

$$\hat{\mathcal{J}}^\mu(x) = \ldots + \frac{2}{3}e\,\hat{\bar{c}}(x)\gamma^\mu\hat{c}(x) - \frac{1}{3}e\,\hat{\bar{b}}(x)\gamma^\mu\hat{b}(x) \;.$$

[Rather than $\tilde{C}_>$ one often considers the spectral function:

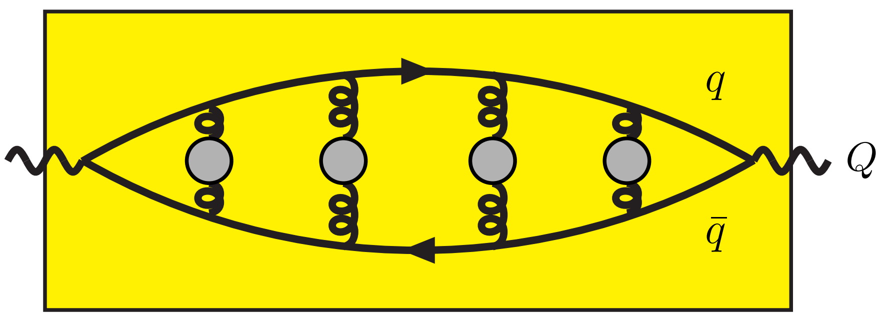
$$\rho(Q) = \frac{1}{2}(1-e^{-\frac{q^0}{T}})\tilde{C}_>(Q) \;.$$

]

Let us consider energies near the two-quark threshold, $q^0 - [2M + \mathbf{q}^2/4M] \ll M$.

Then this corresponds roughly (though not precisely) to a bound state problem generalized to finite temperature.

$\Rightarrow$ Need to resum graphs!



Bound state problem at $T = 0$

Energy scales: $M, g^2M, g^4M, \ldots$

Integrate out $M \Rightarrow$ NRQCD

Caswell, Lepage PLB 167 (1986) 437

Integrate out $g^2M \Rightarrow$ pNRQCD

Pineda, Soto hep-ph/9707481

Brambilla et al hep-ph/9907240

E.g.: ground state energy:

$$E_{nlj} = 2M \left(1 + \#_1 \, g^4 + \#_2 \, g^6 + \#_3 \, g^8 + \#_4 \, g^{10} \ln \frac{1}{g} + \ldots \right) \; .$$

Let us concentrate on the $\mathcal{O}(g^4)$ radiative correction. It comes from the Schrödinger equation,

$$\left[2M - \frac{g^2 C_F}{4\pi r} - \frac{\nabla_{\mathbf{r}}^2}{M}\right] \psi = E\, \psi \ .$$

Scales:

$$|E - 2M| \ll M \quad \Rightarrow \quad \frac{g^2}{r} \sim \frac{1}{r^2 M}$$

$$\Rightarrow \quad p \sim \frac{1}{r} \sim g^2 M$$

$$\Rightarrow \quad E - 2M \sim \frac{p^2}{M} \sim g^4 M \ .$$

What happens after the insertion of $V_>(\infty, r)$?

(a) $T \sim g^2 M$

$\Rightarrow m_{\rm D} r \sim gTr \sim g^3 M r \sim g$

$\Rightarrow \mathrm{Re}\, V_> \sim g^2 \exp(-m_{\rm D} r)/r \sim g^4 M$

$\Rightarrow \mathrm{Im}\, V_> \sim g^2 T (m_{\rm D} r)^2 \sim g^6 M$

$\Rightarrow$ width $\ll$ binding energy $\Rightarrow$ bound state exists.

(b) $T \sim gM$

$\Rightarrow m_{\rm D} r \sim gTr \sim g^2 M r \sim 1$

$\Rightarrow \mathrm{Re}\, V_> \sim g^2 \exp(-m_{\rm D} r)/r \sim g^4 M$

$\Rightarrow \mathrm{Im}\, V_> \sim g^2 T \phi(m_{\rm D} r) \sim g^3 M$

$\Rightarrow$ width $\gg$ binding energy $\Rightarrow$ bound state has melted.

Burnier et al 0711.1743

In order to consider systematically various temperature regimes as well as to include higher-order corrections, need to embed $V_{>}(\infty, r)$ as a matching coefficient in the NRQCD / pNRQCD framework.

Escobedo, Soto 0804.0691; Brambilla et al 0804.0993

However, this does not change anything at the order considered above.

In fact, the parametric estimate concerning melting can be refined into $T_{\text{melt}} \sim g^{4/3} M$.

Escobedo, Soto 0804.0691

3. Physics lessons

(a) Conceptual

Because of the imaginary part, there is no stationary wave function at high temperatures:

$$i\partial_t C_> = \left[2M + \mathrm{Re}\, V_> - i |\, \mathrm{Im}\, V_> | - \frac{\nabla_{\mathbf{r}}^2}{M} \right] C_>(t, r)$$

$\Rightarrow$ exponential decay with time
$\Rightarrow$ the bound state is a short-lived transient.

(b) Practical: ρ and $\frac{\mathrm{d}N_{\mu^+\mu^-}}{\mathrm{d}^4x\mathrm{d}^4Q}$ for $g^2M \lesssim T \lesssim gM$.

Solve

$$i\partial_t C_>(t;\mathbf{r},\mathbf{r}') = \left[2M + V_>(\infty,r) - \frac{\nabla_\mathbf{r}^2}{M} + \mathcal{O}\Big(\frac{1}{M^2}\Big)\right] C_>(t;\mathbf{r},\mathbf{r}')$$

with the initial condition

$$C_>(0;\mathbf{r},\mathbf{r}') = -6N_\mathrm{c}\,\delta^{(3)}(\mathbf{r}-\mathbf{r}') + \mathcal{O}\Big(\frac{1}{M}\Big)\ .$$

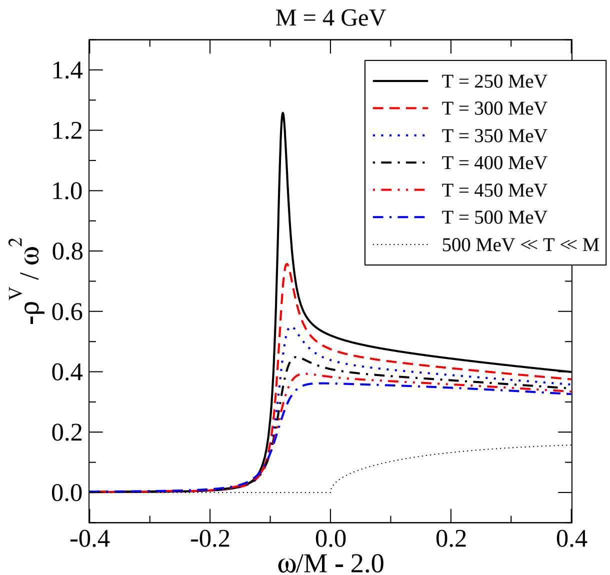
Then

$$\rho(\omega) = \frac{1}{2}\Big(1 - e^{-\beta\omega}\Big)\int_{-\infty}^{\infty}\mathrm{d}t\, e^{i\omega t} C_>(t;\mathbf{0},\mathbf{0})\ .$$

Burnier et al 0711.1743

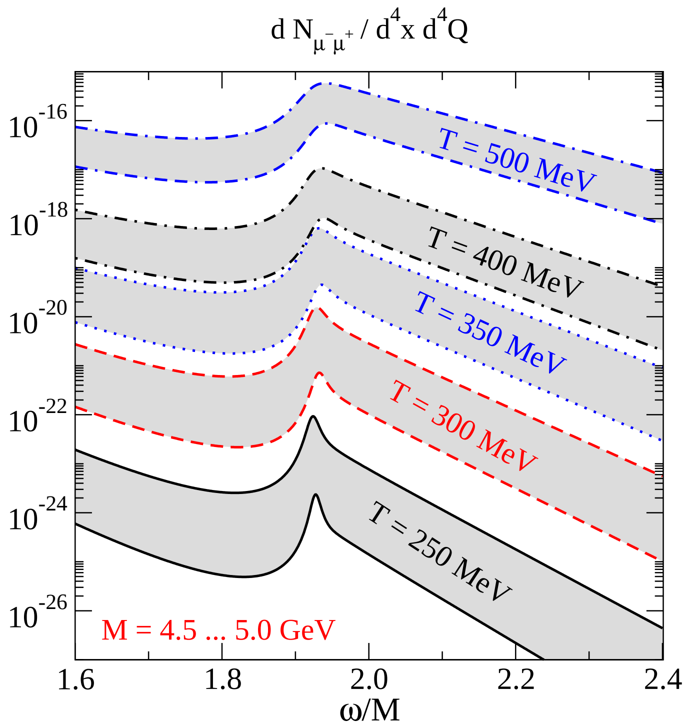
Melting of the spectral fcn in the vector channel:

ML 0704.1720



Basic structure as suggested by Matsui and Satz (1986) from phenomenological arguments.
Melting temperature $\sim$ consistent with potential models and lattice QCD within ± 50 MeV.

But tables turn for dilepton yield! $\frac{\mathrm{d}N_{\mu^+\mu^-}}{\mathrm{d}^4x\,\mathrm{d}^4Q} \propto \frac{\rho(\omega)}{\omega^2} e^{-\frac{\omega}{T}}$



No need to bind in order to produce a structure.

4. Mystery with the scalar channel

($\equiv$ 2pt correlator without γ^μ's; $q\bar{q} \to \mu^-\mu^+\gamma$)

On the level of correlators:

$$C^S_>(t;\mathbf{r},\mathbf{r}') \simeq -\frac{1}{3M^2}\nabla_\mathbf{r}\cdot\nabla_{\mathbf{r}'}\, C^V_>(t;\mathbf{r},\mathbf{r}') + \mathcal{O}\left(\frac{1}{M^3}\right) \; .$$

Schrödinger eqn can be transformed to frequency space:

$$\Big[\omega - \hat{H} + i|\,\mathrm{Im}\,V_>(r)|\Big]\,\tilde{\Psi}(\omega;\mathbf{r},\mathbf{r}') = -6N_\mathrm{c}\delta^{(3)}(\mathbf{r}-\mathbf{r}') \; .$$

Ansatz for the solution:

Strassler, Peskin, PRD 43 (1991) 1500

$$\tilde{\Psi}(\omega;\mathbf{r},\mathbf{r}') \equiv \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{\tilde{g}_l(\omega;r,r')}{rr'}Y_{lm}(\Omega)Y^*_{lm}(\Omega')\ .$$

The spectral functions are now obtained from

$$\begin{aligned}\rho^V(\omega) &= -\lim_{\mathbf{r},\mathbf{r}'\to\mathbf{0}}\mathrm{Im}[\tilde{\Psi}(\omega;\mathbf{r},\mathbf{r}')]\ ,\\ \rho^S(\omega) &\simeq \lim_{\mathbf{r},\mathbf{r}'\to\mathbf{0}}\frac{1}{3M^2}\mathrm{Im}[\nabla_{\mathbf{r}}\cdot\nabla_{\mathbf{r}'}\tilde{\Psi}(\omega;\mathbf{r},\mathbf{r}')]\ .\end{aligned}$$

The solution behaves as

$$\tilde{g}_l \sim [r^{l+1} + \mathcal{O}(r^{l+2})][(r')^{l+1} + \mathcal{O}((r')^{l+2})] \ .$$

For the vector channel,

$$\rho^V(\omega) = - \lim_{r,r' \to 0} \frac{1}{4\pi r r'} \mathrm{Im}[\tilde{g}_0(\omega; r, r')] \ ,$$

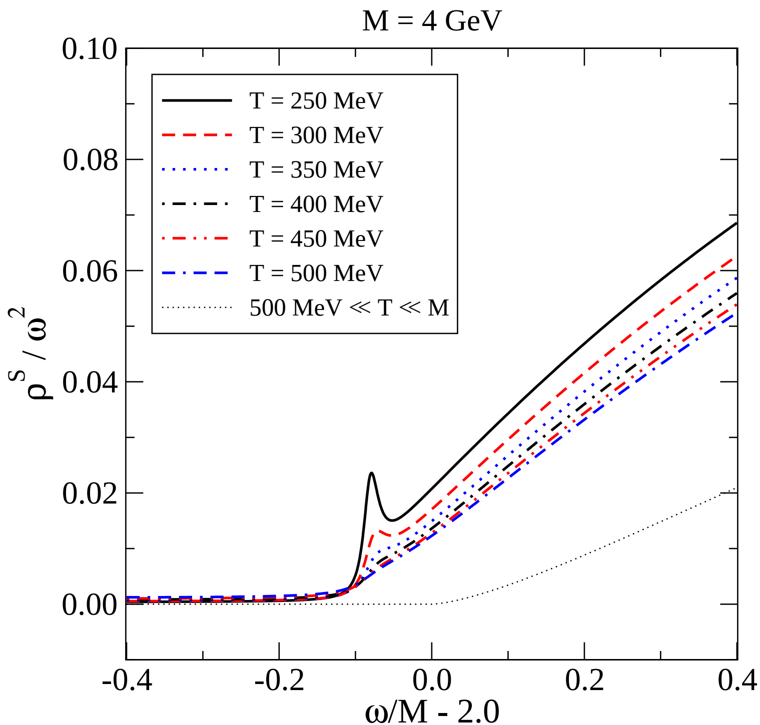
i.e. only the S-wave ($l = 0$) contributes.

For the scalar channel, take two derivatives and extrapolate $r, r' \longrightarrow 0$. Get at least a contribution from the P-wave ($l = 1$), which supports *no resonance peak*.

However, it's also possible to get a contribution from the *subleading S-wave terms*, $\tilde{g}_0 \sim [r + \mathcal{O}(r^2)][r' + \mathcal{O}((r')^2)]$. This yields a peak!

Melting of the spectral fcn in the scalar channel:

Burnier et al 0711.1743



Conclusions

1. Real-time static potential at finite temperature

2. Relation of static potential and spectral function

3. Physics lessons for dilepton production

4. Mystery with the scalar channel

Appendix A: momentum scales at $T \neq 0$

$$\boxed{\text{QCD} \equiv \text{4d YM} + \text{quarks; } \omega_n \sim 2\pi T}$$

$\Downarrow$ perturbation theory (1)

$$\boxed{\text{EQCD} \equiv \text{3d YM} + A_0\text{; } m_\mathrm{D} \sim gT}$$

$\Downarrow$ perturbation theory (2)

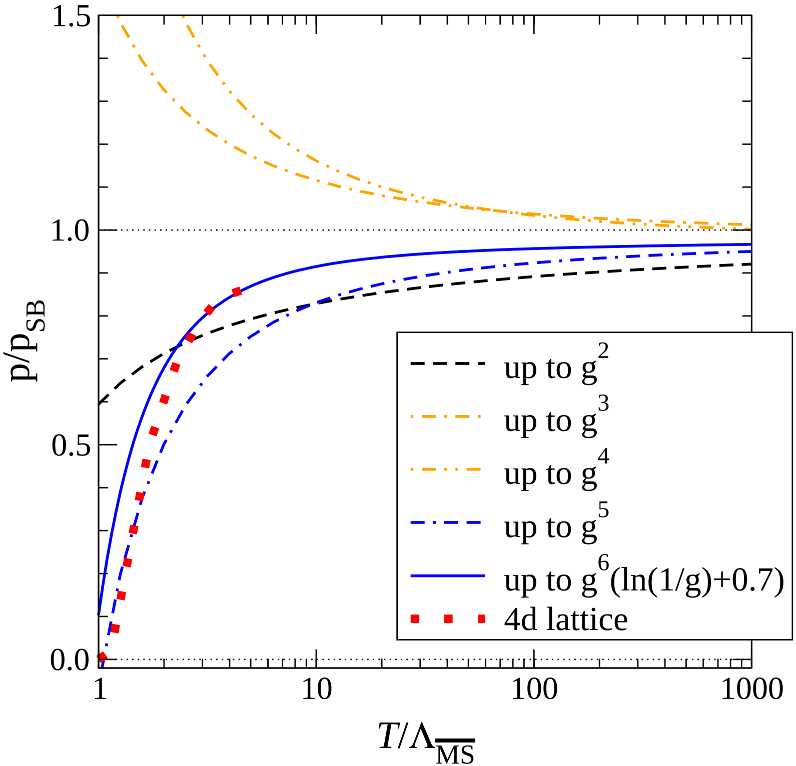
$$\boxed{\text{MQCD} \equiv \text{3d YM; } g_3^2 \sim g^2 T}$$

$\Downarrow$ non-perturbative computation (3)

$$\boxed{\text{PHYSICS}}$$

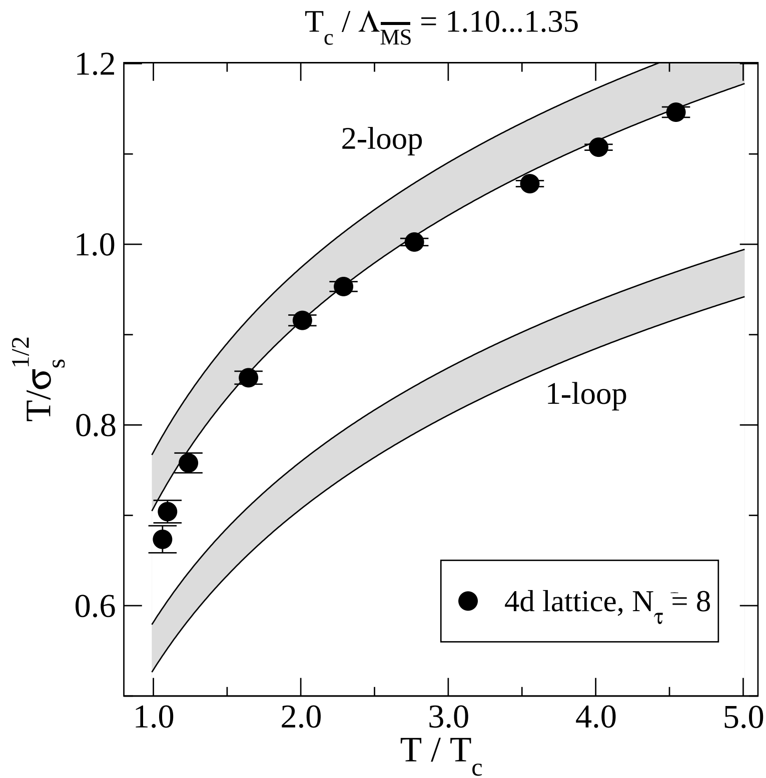
Expansion parameter: $\epsilon_{(i)} \sim g^2 T / 4\pi |\mathbf{k}|_{(i)}$.

Example of slow convergence: pressure



Kajantie et al, hep-ph/0211321

Example of faster convergence: spatial string tension



ML, Schröder, hep-ph/0503061

Appendix B: Time orderings at finite temperature

Consider 2-point functions; $x \equiv (t, \mathbf{x})$; $\tilde{x} \equiv (\tau, \mathbf{x})$;

$\hat{A}(t) = e^{i\hat{H}t}\hat{A}(0)e^{-i\hat{H}t}$; $\hat{A}(\tau) = e^{\hat{H}\tau}\hat{A}(0)e^{-\hat{H}\tau}$.

$$\tilde{C}_{>}(Q) \equiv \int \mathrm{d}t\,\mathrm{d}^3\mathbf{x}\, e^{iQ\cdot x} \langle \hat{A}(x)\hat{B}(0) \rangle \ ,$$

$$\tilde{C}_{<}(Q) \equiv \int \mathrm{d}t\,\mathrm{d}^3\mathbf{x}\, e^{iQ\cdot x} \langle \hat{B}(0)\hat{A}(x) \rangle \ ,$$

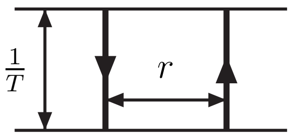
$$\tilde{C}_{R}(Q) \equiv i \int \mathrm{d}t\,\mathrm{d}^3\mathbf{x}\, e^{iQ\cdot x} \langle [\hat{A}(x), \hat{B}(0)]\theta(t) \rangle \ ,$$

$$\tilde{C}_{T}(Q) \equiv \int \mathrm{d}t\,\mathrm{d}^3\mathbf{x}\, e^{iQ\cdot x} \langle \hat{A}(x)\hat{B}(0)\theta(t) + \hat{B}(0)\hat{A}(x)\theta(-t) \rangle ,$$

$$\tilde{C}_{E}(\tilde{Q}) \equiv \int_0^\beta \mathrm{d}\tau \int \mathrm{d}^3\mathbf{x}\, e^{i\tilde{Q}\cdot\tilde{x}} \langle \hat{A}(\tilde{x})\hat{B}(0) \rangle \ .$$

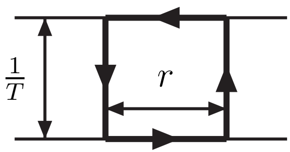
Appendix C: A few different static potentials

From Polyakov loops:



$$\langle \mathrm{Tr}[P]\,\mathrm{Tr}[P^\dagger]\rangle \equiv e^{-\frac{V_\mathsf{a}(r,T)}{T}} \; .$$

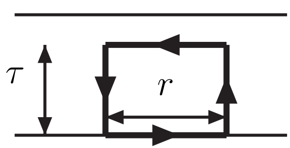
From a Wilson loop:



$$\langle \mathrm{Tr}[W_E(\frac{1}{T}, r)]\rangle \equiv e^{-\frac{V_\mathsf{b}(r,T)}{T}} \; .$$

Or may also Legendre transform from "free energy" to "internal energy": $U_\mathsf{i} = V_\mathsf{i} + TS_\mathsf{i} = V_\mathsf{i} - T\partial_T V_\mathsf{i}$.

From an analytic continuation:



$$\langle \mathrm{Tr}[W_E(\tau, r)]\rangle \equiv C_E(\tau, r) \; .$$

Appendix D: Hard Thermal Loop propagators

Introducing the projection operators

$$P^T_{00}(\tilde{Q}) \;=\; P^T_{0i}(\tilde{Q}) = P^T_{i0}(\tilde{Q}) \equiv 0 \;, \quad P^T_{ij}(\tilde{Q}) \equiv \delta_{ij} - \frac{\tilde{q}_i \tilde{q}_j}{\tilde{\mathbf{q}}^2} \;,$$

$$P^E_{\mu\nu}(\tilde{Q}) \;\equiv\; \delta_{\mu\nu} - \frac{\tilde{q}_\mu \tilde{q}_\nu}{\tilde{Q}^2} - P^T_{\mu\nu}(\tilde{Q}) \;,$$

the Euclidean gluon propagator reads

$$\langle A^a_\mu A^b_\nu \rangle = \delta^{ab} \Big[\frac{P^T_{\mu\nu}(\tilde{Q})}{\tilde{Q}^2 + \Pi_T(\tilde{Q})} + \frac{P^E_{\mu\nu}(\tilde{Q})}{\tilde{Q}^2 + \Pi_E(\tilde{Q})} + \xi \frac{\tilde{q}_\mu \tilde{q}_\nu}{(\tilde{Q}^2)^2} \Big] \;,$$

where ξ is the gauge parameter.

The Hard Thermal Loop self-energies read

$$\Pi_T(\tilde{Q}) \quad = \quad \frac{m_\mathrm{D}^2}{2}\Big\{\frac{(i\tilde{q}_0)^2}{\tilde{\mathbf{q}}^2} + \frac{i\tilde{q}_0}{2|\tilde{\mathbf{q}}|}\Big[1 - \frac{(i\tilde{q}_0)^2}{\tilde{\mathbf{q}}^2}\Big] \ln \frac{i\tilde{q}_0 + |\tilde{\mathbf{q}}|}{i\tilde{q}_0 - |\tilde{\mathbf{q}}|}\Big\} \; ,$$

$$\Pi_E(\tilde{Q}) \quad = \quad m_\mathrm{D}^2\Big[1 - \frac{(i\tilde{q}_0)^2}{\tilde{\mathbf{q}}^2}\Big]\Big[1 - \frac{i\tilde{q}_0}{2|\tilde{\mathbf{q}}|} \ln \frac{i\tilde{q}_0 + |\tilde{\mathbf{q}}|}{i\tilde{q}_0 - |\tilde{\mathbf{q}}|}\Big] \; ,$$

where $\tilde{q}_0$ denotes bosonic Matsubara frequencies, and

$$m_\mathrm{D}^2 = g^2T^2\Big(\frac{N_\mathrm{c}}{3} + \frac{N_\mathrm{f}}{6}\Big) \; .$$

Graphs: